

How to Quantify Graph De-anonymization Risks^{*}

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Abstract. An increasing amount of data are becoming publicly available over the Internet. These data are released after applying some anonymization techniques. Recently, researchers have paid significant attention to analyzing the risks of publishing privacy-sensitive data. Even if data anonymization techniques were applied to protect privacy-sensitive data, several de-anonymization attacks have been proposed to break their privacy. However, no theoretical quantification for relating the data vulnerability against de-anonymization attacks and the data utility that is preserved by the anonymization techniques exists.

In this paper, we first address several fundamental open problems in the structure-based de-anonymization research by establishing a formal model for privacy breaches on anonymized data and quantifying the conditions for successful de-anonymization under a general graph model. To the best of our knowledge, this is the first work on quantifying the relationship between anonymized utility and de-anonymization capability. Our quantification works under very general assumptions about the distribution from which the data are drawn, thus providing a theoretical guide for practical de-anonymization/anonymization techniques.

Furthermore, we use multiple real-world datasets including a Facebook dataset, a Collaboration dataset, and two Twitter datasets to show the limitations of the state-of-the-art de-anonymization attacks. From these experimental results, we demonstrate the ineffectiveness of previous de-anonymization attacks and the potential of more powerful de-anonymization attacks in the future, by comparing the theoretical de-anonymization capability proposed by us with the practical experimental results of the state-of-the-art de-anonymization methods.

Keywords: Structure-based de-anonymization attacks; anonymization utility; de-anonymization capability; theoretical bounds;

1 INTRODUCTION

Individual users' data such as social relationships, medical records and mobility traces are becoming increasingly important for application developers and data-

^{*} This is an extended version of [1] in ICISSP 2017.

mining researchers. These data usually contain sensitive and private information about users. Therefore, several data anonymization techniques have been proposed to protect users' privacy [2], [3], [4].

The privacy-sensitive data that are closely related to individual behavior usually contain rich graph structural characteristics. For instance, social network data can be modeled as graphs in a straightforward manner. Mobility traces can also be modeled as graph topologies according to [5]. Many people nowadays have accounts through various social networks such as Facebook, Twitter, Google+, Myspace and Flickr. Therefore, even equipped with advanced anonymization techniques, the privacy of structural data still suffers from de-anonymization attacks assuming that the adversaries have access to rich auxiliary information from other channels [5], [6], [7], [8], [9], [10], [11]. For instance, Narayanan et al. [8] effectively de-anonymized a Twitter dataset by utilizing a Flickr dataset as auxiliary information based on the inherent cross-site correlations. Nilizadeh et al. [10] exploited the community structure of graphs to de-anonymize social networks. Furthermore, Srivatsa et al. [5] proposed to de-anonymize a set of location traces based on a social network.

However, to the best of our knowledge, there is no work on theoretically quantifying the data anonymization techniques to defend against de-anonymization attacks. In this paper, we aim to theoretically analyze the de-anonymization attacks in order to provide effective guidelines for evaluating the threats of future de-anonymization attacks. We aim to rigorously evaluate the vulnerabilities of existing anonymization techniques. For an anonymization approach, not only the users' sensitive information should be protected, but also the anonymized data should remain useful for applications, i.e., the *anonymized utility* should be guaranteed. Then, under what range of anonymized utility, is it possible for the privacy of an individual to be broken? We will quantify the vulnerabilities of existing anonymization techniques and establish the inherent relationships between the application-specific anonymized utility and the de-anonymization capability. Our quantification not only provides theoretical foundations for existing de-anonymization attacks, but also can serve as a guide for designing new de-anonymization and anonymization schemes. For example, the comparison between the theoretical de-anonymization capability and the practical experimental results of current de-anonymization attacks demonstrates the ineffectiveness of existing de-anonymization attacks. Overall, we make the following contributions:

- We theoretically analyze the performance of structure-based de-anonymization attacks through formally quantifying the vulnerabilities of anonymization techniques. Furthermore, we rigorously quantify the relationships between the de-anonymization capability and the utility of anonymized data, which is the first such attempt to the best of our knowledge. Our quantification provides theoretical foundations for existing structure-based de-anonymization attacks, and can also serve as a guideline for evaluating the effectiveness of new de-anonymization and anonymization schemes through comparing their corresponding de-anonymization performance with our derived theoretical bounds.

- To demonstrate the ineffectiveness of existing de-anonymization attacks, we implemented these attacks on multiple real-world datasets including Facebook dataset, Collaboration dataset, and Twitter dataset. Experimental results show that previous methods are not robust to data perturbations and there is a significant gap between their de-anonymization performance and our derived theoretically achievable de-anonymization capability. This analysis further demonstrates the potential of developing more powerful de-anonymization attacks in the future.

2 RELATED WORK

2.1 Challenges for Anonymization Techniques

Privacy preservation on structural data has been studied extensively. The naive method is to remove users' personal identities (e.g., names, social security numbers), which, unfortunately, is rather vulnerable to structure-based de-anonymization attacks [6], [7], [8], [12], [3], [5], [9], [13], [14], [10], [15]. An advanced mechanism, k -anonymity, was proposed in [12], which obfuscates the attributes of users so that each user is indistinguishable from at least $k - 1$ other users. Although k -anonymity has been well adopted, it still suffers from severe privacy problems due to the lack of diversity with respect to the sensitive attributes as stated in [16]. Differential privacy [17], [18] is a popular privacy metric that statistically minimizes the privacy leakage. Sala et al. in [19] proposed to share a graph in a differentially private manner. However, to enable the applicability of such an anonymized graph, the differential private parameter should not be large, which would thus make their method ineffective in defending against structure-based de-anonymization attacks [9]. Hay et al. in [2] proposed a perturbation algorithm that applies a sequence of r random edge deletions followed by r other random edge insertions. However, their method also suffers from structure-based de-anonymization attacks as shown in [10].

In summary, existing anonymization techniques are subject to two intrinsic limitations: 1) they are not scalable and thus would fail on high-dimensional datasets; 2) They are susceptible to adversaries that leverage the rich amount of auxiliary information to achieve structure-based de-anonymization attacks.

2.2 De-anonymization Techniques

Structure-based de-anonymization was first introduced in [6], where both active and passive attacks were discussed. However, the limitation of scalability reduces the effectiveness of both attacks.

Narayanan et al. in [7] utilized the Internet movie database as the source of background knowledge to successfully identify users' Netflix records, uncovering their political preferences and other potentially sensitive information. In [8], Narayanan et al. further de-anonymized a Twitter dataset using a Flickr dataset as auxiliary information. They proposed the popular seed identification

and mapping propagation process for de-anonymization. In order to obtain the seeds, they assume that the attacker has access to a small number of members of the target network and can determine if these members are also present in the auxiliary network (e.g., by matching user names and other contextual information). Srivatsa et al. in [5] captured the WiFi hotspot and constructed a contact graph by connecting users who are likely to utilize the same WiFi hotspot for a long time. Based on the fact that friends (or people with other social relationships) are likely to appear in the same location, they showed how mobility traces can be de-anonymized using an auxiliary social network. However, their de-anonymization approach is rather time-consuming and may be computationally infeasible for real-world applications. In [14], [13], Sharad et al. studied the de-anonymization attacks on ego graphs with graph radius of one or two and they only studied the linkage of nodes with degree greater than five. As shown in previous work [9], nodes with degree less than five cannot be ignored since they form a large portion of the original real-world data. Recently, Nilizadeh et al. [10] proposed a community-enhanced de-anonymization scheme for social networks. The community-level de-anonymization is first implemented for finding more seed information, which would be leveraged for improving the overall de-anonymization performance. Their method may, however, suffer from the serious inconsistency problem of community detection algorithms [20].

Most de-anonymization attacks are based on the seed-identification scheme, which either relies on the adversary’s prior knowledge or a seed mapping process. Limited work has been proposed that requires no prior seed knowledge by the adversary [21],[9]. In [21], Pedarsani et al. proposed a Bayesian-inference approach for de-anonymization. However, their method is limited to de-anonymizing sparse graphs. Ji et al. in [9] proposed a cold-start optimization-based de-anonymization attack. However, they only utilized very limited structural information (degree, neighborhood, top-K reference distance and sampling closeness centrality) of the graph topologies.

Ji et al. further made a detailed comparison for the performance of existing de-anonymization techniques in [22].

2.3 Theoretical Work for De-anonymization

Despite these empirical de-anonymization methods, limited research has provided theoretical analysis for such attacks. Pedarsani et al. in [4] conducted preliminary analysis for quantifying the privacy of an anonymized graph according to the ER graph model [23]. However, their network model (ER model) may not be realistic, since the degree distribution of the ER model (follows the Poisson distribution) is quite different from the degree distributions of most observed real-world structural data [24], [25].

Ji et al. in [9] further considered a configuration model to quantify perfect de-anonymization and $(1 - \epsilon)$ -perfect de-anonymization. However, their configuration model is also not general for many real-world data structures. Furthermore, their assumption that the anonymized and the auxiliary graphs are sampled from a conceptual graph is not practical since only edge deletions from

the conceptual graph have been considered. In reality, edge insertions should also be taken into consideration. Besides, neither [4] nor [9] formally analyzed the relationships between the de-anonymization capability and the anonymization performance (e.g., the utility performance for the anonymization schemes).

Note that our theoretical analysis in Section 4 takes the application-specific utility definition into consideration. Such non-linear utility analysis makes the incorporation of edge insertions to our quantification rather nontrivial. Furthermore, our theoretical quantification does not make any restrictive assumptions about the graph model. Therefore, our theoretical analysis would provide an important guide for relating de-anonymization capability and application-specific anonymizing utility.

Further study on de-anonymization attacks can be found in [26], [27], [28]. These papers provide theoretically guaranteed performance bounds for their de-anonymization algorithms. However, their derived performance bounds can only be guaranteed under restricted assumptions of the random graph, such as ER model and power-law model. We will show the advantage of our analysis over these approaches where our analysis requires no assumptions or constraints on the graph model as these approaches required.

3 SYSTEM MODEL

We model the structural data (e.g., social networks, mobility traces, etc.) as a graph, where the nodes represent users who are connected by certain relationships (social relationships, mobility contacts, etc.). The anonymized graph can be modeled as $G_a = (V_a, E_a)$, where $V_a = \{i | i \text{ is an anonymized node}\}$ is the set of users and $E_a = \{e_a(i, j) | e_a(i, j) \text{ is the relationship between } i \in V_a \text{ and } j \in V_a\}$ is the set of relationships. Here, $e_a(i, j) = 1$ represents the existence of a connecting edge between i and j in G_a , and $e_a(i, j) = 0$ represents the non-existence of such an edge. The neighborhood of node $i \in V_a$ is $N_a(i) = \{j | e_a(i, j) = 1\}$ and the degree is defined as $|N_a(i)|$.

Similarly, the auxiliary structural data can also be modeled as a graph $G_u = (V_u, E_u)$ where V_u is the set of labeled (known) users and E_u is the set of relationships between these users. Note that the auxiliary (background) data can be easily obtained through various channels, e.g., academic data mining, online crawling, advertising and third-party applications [8, 4, 29, 5].

A de-anonymization process is a mapping $\sigma : V_a \rightarrow V_u$. $\forall i \in V_a$, its mapping under σ is $\sigma(i) \in V_u \cup \{\perp\}$, where \perp indicates a non-existent (null) node. Similarly, $\forall e_a(i, j) \in E_a$, $\sigma(e_a(i, j)) = e_u(\sigma(i), \sigma(j)) \in E_u \cup \{\perp\}$. Under σ , a successful de-anonymization on $i \in V_a$ is defined as $\sigma(i) = i$, if $i \in V_u$ or $\sigma(i) = \perp$, otherwise. For other cases, the de-anonymization on i fails.

3.1 ATTACK MODEL

We assume that the adversary has access to $G_a = (V_a, E_a)$ and $G_u = (V_u, E_u)$. $G_a = (V_a, E_a)$ is the anonymized graph and the adversary can only get access

to the structural information of G_a . $G_u = (V_u, E_u)$ is the auxiliary graph and the adversary already knows all the identities of the nodes in G_u . In addition, we do not assume that the adversary has other prior information (e.g., seed information). These assumptions are more reasonable than most of the state-of-the-art research [8, 5, 10].

4 THEORETICAL ANALYSIS

In this section, we provide a theoretical analysis for the structure-based de-anonymization attacks. Under any anonymization technique, the users' sensitive information should be protected without significantly affecting the utility of the anonymized data for real-world systems or research applications. We aim to quantify the trade-off between preserving users' privacy and the utility of anonymized data. Under what range of anonymized utility, is it possible for the privacy of an individual to be broken (i.e., for the success of de-anonymization attacks)? To answer this, we quantify the limitations of existing anonymization schemes and establish an inherent relationship between the anonymized utility and de-anonymization capability. Our theoretical analysis incorporates an application-specific utility metric for the anonymized graph, which further makes our rigorous quantification useful for real world scenarios. Our theoretical analysis can serve as an effective guideline for evaluating the performance of practical de-anonymization/anonymization schemes (will be discussed in Section 6).

First, we assume that there exists a conceptually underlying graph $G = (V, E)$ with $V = V_a \cup V_u$ and E is a set of relationships among users in V , where $e(i, j) = 1 \in E$ represents the existence of a connecting edge between i and j , and $e(i, j) = 0 \in E$ represents the non-existence of such an edge. Consequently, G_a and G_u could be viewed as observable versions of G by applying edge insertions or deletions on G according to proper relationships, such as 'co-occurrence' relationships in Gowalla dataset [29]. In comparison, previous work [9, 4] only considers edge deletions which is an unrealistic assumption.

For edge insertions from G to G_a , the process is: $\forall e(i, j) = 0 \in E$, $e(i, j) = 1$ appears in E_a with probability p_a^{add} , i.e., $Pr(e_a(i, j) = 1 | e(i, j) = 0) = p_a^{\text{add}}$. The probability of edge deletion from G to G_a is p_a^{del} , i.e., $Pr(e_a(i, j) = 0 | e(i, j) = 1) = p_a^{\text{del}}$. Similarly, the insertions and deletions from G to G_u can be characterized with probabilities p_u^{add} and p_u^{del} . Furthermore, we assume that both the insertion/deletion relationship of each edge is independent of every other edge. Furthermore, this model is intuitively reasonable since the three graphs G , G_a , G_u are related with each other. In addition, our model is more reasonable than the existing models in [9, 4] because we take both edge deletions and insertions into consideration. Note that the incorporation of edge insertion is non-trivial in our quantification of non-linear application-specific utility analysis. Our quantification analysis would therefore contribute to relating the real world application-specific anonymizing utility and the de-anonymization capability.

Adjacency matrix and transition probability matrix are two important des-

criptions of a graph, and the graph utility is also closely related to these matrices. The adjacency matrix is a means of representing which nodes of a graph are adjacent to which other nodes. We denote the adjacency matrix by \mathbf{A} (resp. \mathbf{A}_a and \mathbf{A}_u) for graph G (resp. G_a and G_u), where the element $\mathbf{A}(i, j) = e(i, j)$ (resp. $\mathbf{A}_a(i, j) = e_a(i, j)$ and $\mathbf{A}_u(i, j) = e_u(i, j)$). Furthermore, the transition probability matrix is a matrix consisting of the one-step transition probabilities, which is the probability of transitioning from one node to another in a single step. We denote the transition probability matrix by \mathbf{T} (resp. \mathbf{T}_a and \mathbf{T}_u) for graph G (resp. G_a and G_u), where the element $\mathbf{T}(i, j) = e(i, j)/deg(i)$ (resp. $\mathbf{T}_a(i, j) = e_a(i, j)/deg_a(i)$ and $\mathbf{T}_u(i, j) = e_u(i, j)/deg_u(i)$), and $deg(i), deg_a(i), deg_u(i)$ represent the degree of node i in G, G_a, G_u , respectively.

We now define the smallest (l) and largest (h) probabilities of an edge existing between two nodes in the graph G , and the graph density (denoted by R). For graph G , we denote $|V| = N$ and $|E| = M$. Let $p(i, j)$ be the probability of an edge existing between $i, j \in V$ and define $l = \min\{p(i, j) | i, j \in V, i \neq j\}$, $h = \max\{p(i, j) | i, j \in V, i \neq j\}$, the expected number of edges $P_T = \sum_{i, j \in V} p(i, j)$ and the graph density $R = \frac{P_T}{\binom{N}{2}}$.

Then, we start our formal quantification from the simplest scenario where the anonymized data and the auxiliary data correspond to the same group of users i.e., $V_a = V_u$ as in [8, 4, 5]. This assumption does not limit our theoretical analysis since we can either (a) apply it to the overlapped users between V_a and V_u or (b) extend the set of users to $V_a^{new} = V_a \cup (V_u \setminus V_a)$ and $V_u^{new} = V_u \cup (V_a \setminus V_u)$, and apply the analysis to $G_a = (V_a^{new}, E_a)$ and $G_u = (V_u^{new}, E_u)$. Therefore, in order to prevent any confusion and without loss of generality, we assume $V_a = V_u$ in our theoretical analysis. We define σ_k as a mapping between G_a and G_u that contains k incorrectly-mapped pairs.

Given a mapping $\sigma : V_a \rightarrow V_u$, we define the *Difference of Common Neighbors (DCN)* on a node i 's mapping $\sigma(i)$ as $\phi_{i, \sigma(i)} = |N_a^i \setminus N_u^{\sigma(i)}| + |N_u^{\sigma(i)} \setminus N_a^i|$, which measures the neighborhoods' difference between node i in G_a and node $\sigma(i)$ in G_u under the mapping σ . Then, we define the overall *DCN* for all the nodes under the mapping σ as $\Phi_\sigma = \sum_{(i, \sigma(i)) \in \sigma} \phi_{i, \sigma(i)}$.

Next, we not only explain why structure-based de-anonymization attacks work but also quantify the trade-off between the anonymized utility and the de-anonymization capability. We first quantify the relationship between a straightforward utility metric, named *local neighborhood utility*, and the de-anonymization capability. Then we carefully analyze a more general utility metric, named *global structure utility*, to accommodate a broad class of real-world applications.

4.1 Relation Between the Local Neighborhood Utility and De-anonymization Capability

At the beginning, we explore a straightforward utility metric, local neighborhood utility, which evaluates the distortion of the anonymized graph G_a from the conceptually underlying graph G as

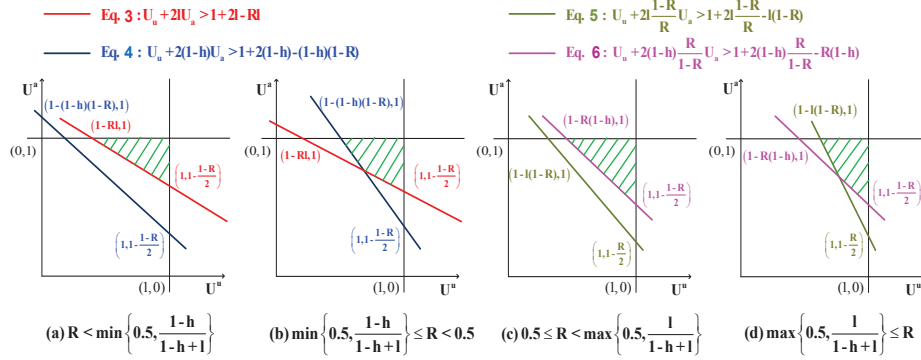


Fig. 1. Visualization of utility region (green shaded) for successful de-anonymization under different scenarios. To guarantee the applicability of the anonymized data, the anonymized utility should be preserved by the anonymization techniques. We theoretically demonstrate that successful de-anonymization can be achieved if the anonymized utility locates within these shaded regions [1].

Definition 1 *The local neighborhood utility for the anonymized graph is $U_a = 1 - \frac{\|\mathbf{A}_a - \mathbf{A}\|_1}{N(N-1)} = 1 - \frac{\mathbb{E}[D(G_a, G)]}{N(N-1)}$ (the denominator is a normalizing factor to guarantee $U_a \in [0, 1]$), where $D(\cdot, \cdot)$ is the hamming distance [30] of edges between two graphs, i.e., if $e_a(i, j) \neq e(i, j)$, $D(e_a(i, j), e(i, j)) = 1$ and $\mathbb{E}[D(G_a, G)]$ is the distortion between G_a and G and $\mathbb{E}[D(G_a, G)] = \mathbb{E}[\sum_{i,j} D(e_a(i, j), e(i, j))] = \sum_{i,j} (p(i, j)p_a^{\text{del}} + (1 - p(i, j))p_a^{\text{add}})$.*

Thus, we further have

$$\begin{aligned}
 U_a &= 1 - \frac{\sum_{i,j} (p(i, j)p_a^{\text{del}} + (1 - p(i, j))p_a^{\text{add}})}{\binom{N}{2}} \\
 &= 1 - (Rp_a^{\text{del}} + (1 - R)p_a^{\text{add}}) [1]
 \end{aligned} \tag{1}$$

Similarly, the local neighborhood utility for the auxiliary graph is

$$U_u = 1 - (Rp_u^{\text{del}} + (1 - R)p_u^{\text{add}}) [1] \tag{2}$$

Though the utility metric for structural data is application-dependent, our utility metric can provide a comprehensive understanding for utility performance by considering both the edge insertions and deletions, and incorporating the distance between the anonymized (auxiliary) graph and the conceptual underlying graph. Although our utility is one of the most straightforward definitions, to the best of our knowledge, it is still the first scientific work that theoretically analyzes the relationship between de-anonymization performance and the utility of the anonymized data. Furthermore, we will provide more analysis by considering a general utility metric that can be applied to a broad coverage of applications.

Based on the local neighborhood utility in Definition 1, we theoretically analyze the de-anonymization capability of structure-based attacks and quantify the anonymized utility for successful de-anonymization.

Theorem 1 implies that as the number of nodes in the graphs G_a and G_u increase, the probability of successful deanonymization approaches 1 when the four conditions (in Eqs. 3,4,5,6) regarding graph density R , and the smallest and largest probabilities of the edges between nodes hold.

Theorem 1 *For any $\sigma_k \neq \sigma_0$, where k is the number of incorrectly-mapped nodes between G_a and G_u , $\lim_{n \rightarrow \infty} Pr(\Phi_{\sigma_k} \geq \Phi_{\sigma_0}) = 1$ when the following conditions are satisfied.*

$$U_u + 2lU_a > 1 + 2l - Rl \quad [1] \quad (3)$$

$$U_u + 2(1-h)U_a > 1 + 2(1-h) - (1-h)(1-R) \quad [1] \quad (4)$$

$$U_u + 2l \frac{1-R}{R} U_a > 1 + 2l \frac{1-R}{R} - l(1-R) \quad [1] \quad (5)$$

$$U_u + 2(1-h) \frac{R}{1-R} U_a > 1 + 2(1-h) \frac{R}{1-R} - R(1-h) \quad [1] \quad (6)$$

From Theorem 1, we know that when the local neighborhood utility for the anonymized graph and the auxiliary graph satisfies the four conditions in Eqs. 3,4,5,6, we can achieve successful de-anonymization *from a statistical perspective*. The reason is that, the attacker can discover the correct mapping with high probability by choosing the mapping with the minimal Difference of Common Neighbors (DCN), out of all the possible mappings between the anonymized graph and the auxiliary graph. To the best of our knowledge, this is the first work to quantify the relationship between anonymized utility and de-anonymization capability. It also essentially explains why structure-based de-anonymization attacks work.

The four conditions in Theorem 1 can be reduced to one or two conditions under four types of graph density. Figure 1(a) is the triangular utility region for $R < \min\{0.5, \frac{1-h}{1-h+l}\}$ (where the graph density R is smaller than 0.5 and $\frac{1-h}{1-h+l}$), which is only bounded by Eq. 3. Figure 1(b) is the quadrilateral utility region for $\min\{0.5, \frac{1-h}{1-h+l}\} \leq R < 0.5$ (where the graph density R is larger than $\frac{1-h}{1-h+l}$ and smaller than 0.5), which is bounded by Eq. 3 and Eq. 4. Similarly, Figure 1(c) is the triangular utility region for $0.5 \leq R < \max\{0.5, \frac{l}{1-h+l}\}$ (where the graph density R is larger than $\frac{l}{1-h+l}$ and 0.5), which is only bounded by Eq. 6. Figure 1(d) is the quadrilateral utility region for $R \geq \max\{0.5, \frac{l}{1-h+l}\}$ (where the graph density R is larger than 0.5 and smaller than $\frac{l}{1-h+l}$), which is bounded by Eq. 5 and Eq. 6. Therefore, we not only analytically explain why the structure-based de-anonymization works, but also theoretically provide the bound of anonymized utility for successful de-anonymization. When the anonymized utility satisfies the conditions in Theorem 1 (or locates within the green shaded utility regions shown in Figure 1), successful de-anonymization is theoretically achievable.

5 Proof for Theorem 1

We first prove the following two lemmas based on the four conditions in Eqs. 3,4,5,6, before obtaining the final result $\lim_{n \rightarrow \infty} Pr(\Phi_{\sigma_k, E_k^u \setminus E_\tau^u} \geq \Phi_{\sigma_0, E_k^u \setminus E_\tau^u}) = 1$ in Theorem 1. The two Lemmas provide the important properties of p_{add}^u, p_{del}^u and $p_{add}^{ua}, p_{del}^{ua}$ respectively, where p_{add}^u, p_{del}^u represent the edge insertion and deletion probabilities from the conceptually underlying graph to the auxiliary graph and $p_{add}^{ua}, p_{del}^{ua}$ represent the edge insertion and deletion probabilities from the auxiliary graph to the anonymized graph.

5.1 Properties for p_{add}^u, p_{del}^u

Lemma 1 Consider $p_{add}^u(i, j)$ which represents the edge insertion probability from G to G^a and $p_{del}^u(i, j)$ which represents the deletion process from G to G^a . We have the following properties for p_{add}^u, p_{del}^u as:

$$\max\{p_{add}^u, p_{del}^u\} < \frac{1}{2} \quad (7)$$

Proof. From Eqs. 3,4,5,6, we can obtain $U^u > \max\{1 - lR, 1 - (1 - h)R, 1 - l(1 - R), 1 - (1 - h)(1 - R)\}$. Based on that, we further have $Rp_{del}^u + (1 - R)p_{add}^u = 1 - U^u < \min\{lR, (1 - h)R, l(1 - R), (1 - h)(1 - R)\} \Rightarrow \frac{1}{2} > \max\{p_{add}^u, p_{del}^u\}$. Similarly, for the anonymized graph, we can prove $\max\{p_{add}^a, p_{del}^a\} < \frac{1}{2}$.

5.2 Properties for $p_{add}^{ua}(i, j), p_{del}^{ua}(i, j)$

Lemma 2 Consider $p_{add}^{ua}(i, j)$ which represents the edge insertion probability from G^u to G^a and $p_{del}^{ua}(i, j)$ which represents the deletion process from G^u to G^a . We have the following properties for $p_{add}^{ua}(i, j), p_{del}^{ua}(i, j)$ as:

$$\max\{p_{add}^{ua}(i, j), p_{del}^{ua}(i, j)\} < \frac{1}{2} \quad (8)$$

Proof. Based on the definition of $p_{add}^{ua}(i, j)$ and $p_{del}^{ua}(i, j)$, we have

$$\begin{aligned} p_{add}^{ua}(i, j) &= Pr(e_{i,j}^a = 1 | e_{i,j}^u = 0) \\ &= \frac{Pr(e_{i,j}^a = 1 | e_{i,j} = 0)Pr(e_{i,j}^u = 0 | e_{i,j} = 0)Pr(e_{i,j} = 0)}{Pr(e_{i,j}^u = 0)} \\ &+ \frac{Pr(e_{i,j}^a = 1 | e_{i,j} = 1)Pr(e_{i,j}^u = 0 | e_{i,j} = 1)Pr(e_{i,j} = 1)}{Pr(e_{i,j}^u = 0)} \\ &= \frac{p_{add}^a(1 - p_{add}^u)(1 - p(i, j)) + (1 - p_{del}^a)p_{del}^u p(i, j)}{(1 - p_{add}^u)(1 - p(i, j)) + p_{del}^u p(i, j)} \end{aligned} \quad (9)$$

$$\begin{aligned}
 p_{\text{del}}^{ua}(i, j) &= Pr(e_{i,j}^a = 0 | e_{i,j}^u = 1) \\
 &= \frac{Pr(e_{i,j}^a = 0 | e_{i,j} = 0) Pr(e_{i,j}^u = 1 | e_{i,j} = 0) Pr(e_{i,j} = 0)}{Pr(e_{i,j}^u = 1)} \\
 &+ \frac{Pr(e_{i,j}^a = 0 | e_{i,j} = 1) Pr(e_{i,j}^u = 1 | e_{i,j} = 1) Pr(e_{i,j} = 1)}{Pr(e_{i,j}^u = 1)} \quad (10) \\
 &= \frac{(1 - p_{\text{add}}^a) p_{\text{add}}^u (1 - p(i, j)) + p_{\text{del}}^a (1 - p_{\text{del}}^u) p(i, j)}{p_{\text{add}}^u (1 - p(i, j)) + (1 - p_{\text{del}}^u) p(i, j)}
 \end{aligned}$$

Next, it is easy to show that Eq. 8 is equivalent to Eqs. 11,12 as follows.

$$(1 - p_{\text{add}}^u)(1 - 2p_{\text{add}}^a)(1 - p(i, j)) > p_{\text{del}}^u(1 - 2p_{\text{del}}^a)p(i, j) \quad (11)$$

$$p_{\text{add}}^u(1 - 2p_{\text{add}}^a)(1 - p(i, j)) < (1 - p_{\text{del}}^u)(1 - 2p_{\text{del}}^a)p(i, j) \quad (12)$$

We prove Eq. 11 and Eq. 12 under the following four situations: (a) $p_{\text{del}}^u \geq p_{\text{add}}^u$ and $p_{\text{del}}^a \geq p_{\text{add}}^a$; (b) $p_{\text{del}}^u \geq p_{\text{add}}^u$ and $p_{\text{del}}^a \leq p_{\text{add}}^a$; (c) $p_{\text{del}}^u \leq p_{\text{add}}^u$ and $p_{\text{del}}^a \geq p_{\text{add}}^a$ and (d) $p_{\text{del}}^u \leq p_{\text{add}}^u$ and $p_{\text{del}}^a \leq p_{\text{add}}^a$, respectively.

Situation (a): under situation (a), we only need to consider Eq. 3 and Eq. 6. Eq. 6 is equivalent to $\frac{1-U^u}{R} < (1-h)(1-2\frac{1-U^a}{R}) \Rightarrow \frac{1-U^u}{R} < (1-h) \Rightarrow \frac{Rp_{\text{del}}^u + (1-R)p_{\text{add}}^u}{R} < 1-h \Rightarrow p_{\text{del}}^u < 1-h$. Based on that, we have $p_{\text{del}}^u < 1-p(i, j)$ and $p(i, j) < 1-p_{\text{del}}^u < 1-p_{\text{add}}^u$. In addition, we have $1-2p_{\text{del}}^a < 1-2p_{\text{add}}^a$. Therefore, Eq. 11 is satisfied.

Eq. 3 is equivalent to $\frac{1-U^u}{R} < l-2l\frac{1-U^a}{R} \Rightarrow \frac{Rp_{\text{del}}^u + (1-R)p_{\text{add}}^u}{R} < l-2l\frac{Rp_{\text{del}}^a + (1-R)p_{\text{add}}^a}{R} \Rightarrow \frac{Rp_{\text{del}}^u}{R} < l-2l\frac{Rp_{\text{del}}^a}{R} \Rightarrow p_{\text{del}}^u < l(1-2p_{\text{del}}^a) \Rightarrow p_{\text{del}}^u < l(\frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a})$. Therefore, we have

$$\begin{aligned}
 \frac{1}{p_{\text{del}}^u} - 1 &> \frac{1}{l\frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a}} - 1 \\
 &> \frac{1}{p(i, j)\frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a}} - 1 \quad (13) \\
 &= \left(\frac{1}{p(i, j)} - \frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a}\right) \frac{1-2p_{\text{add}}^a}{1-2p_{\text{del}}^a} \\
 &= \left(\frac{1}{p(i, j)} - 1\right) \frac{1-2p_{\text{add}}^a}{1-2p_{\text{del}}^a}
 \end{aligned}$$

Furthermore, we can prove that Eq. 13 is equivalent to

$$(1 - p_{\text{del}}^u)(1 - 2p_{\text{del}}^a)p(i, j) > p_{\text{del}}^u(1 - 2p_{\text{add}}^a)(1 - p(i, j)) \quad (14)$$

Considering that $p_{\text{del}}^u > p_{\text{add}}^u$, Eq. 12 thus holds. Therefore, we have proved that $p_{\text{add}}^ua(i, j) < \frac{1}{2}$ and $p_{\text{del}}^ua(i, j) < \frac{1}{2}$ under situation (a).

Situation (b): under situation (b), we only need to consider Eq. 3 and Eq. 6. Eq. 3 is equivalent to $\frac{1-U^u}{R} < l(1-2\frac{1-U^a}{R}) \Rightarrow \frac{1-U^u}{R} < l \Rightarrow \frac{Rp_{\text{del}}^u + (1-R)p_{\text{add}}^u}{R} < l \Rightarrow p_{\text{del}}^u < l$. Therefore, we can obtain $p_{\text{add}}^u < p_{\text{del}}^u < p(i, j)$ and $1-p(i, j) < 1-p_{\text{del}}^u$. Further combining with $1-2p_{\text{del}}^a < 1-2p_{\text{add}}^a$, we have that Eq. 12 is satisfied.

Eq. 6 is equivalent to $\frac{1-U^u}{R} < (1-h)(1-2\frac{1-U^a}{1-R}) \Rightarrow \frac{Rp_{\text{del}}^u+(1-R)p_{\text{add}}^u}{R} < (1-h)(1-2\frac{Rp_{\text{del}}^a+(1-R)p_{\text{add}}^a}{1-R}) \Rightarrow \frac{Rp_{\text{del}}^u}{R} < (1-h)(1-2\frac{(1-R)p_{\text{add}}^a}{1-R}) \Rightarrow p_{\text{del}}^u < (1-h)(1-2p_{\text{add}}^a) \Rightarrow p_{\text{del}}^u < (1-h)(\frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a})$. Therefore, we have

$$\begin{aligned} \frac{1}{p_{\text{del}}^u} - 1 &> \frac{1}{(1-h)\frac{1-2p_{\text{add}}^a}{1-2p_{\text{del}}^a}} - 1 \\ &> \frac{1}{(1-p(i,j))\frac{1-2p_{\text{add}}^a}{1-2p_{\text{del}}^a}} - 1 \\ &= \left(\frac{1}{(1-p(i,j))} - \frac{1-2p_{\text{add}}^a}{1-2p_{\text{del}}^a}\right) \frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a} \\ &= \left(\frac{1}{(1-p(i,j))} - 1\right) \frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a} \end{aligned} \quad (15)$$

Furthermore, we can prove that Eq. 15 is equivalent to

$$(1-p_{\text{del}}^u)(1-2p_{\text{add}}^a)(1-p(i,j)) > p_{\text{del}}^u(1-2p_{\text{del}}^a)p(i,j) \quad (16)$$

In addition, we have $p_{\text{del}}^u > p_{\text{add}}^u$, and Eq. 11 is thus satisfied. Therefore, we have proved that $p_{\text{add}}^{ua}(i,j) < \frac{1}{2}$ and $p_{\text{del}}^{ua}(i,j) < \frac{1}{2}$ under situation (b).

Situation (c): under situation (c), we only need to consider Eq. 4 and Eq. 5. Eq. 4 is equivalent to $\frac{1-U^u}{1-R} < (1-h)(1-2\frac{1-U^a}{1-R}) \Rightarrow \frac{1-U^u}{1-R} < 1-h \Rightarrow \frac{Rp_{\text{del}}^u+(1-R)p_{\text{add}}^u}{1-R} < 1-h \Rightarrow p_{\text{del}}^u < p_{\text{add}}^u < 1-h$. Therefore, we can obtain $p_{\text{add}}^u < 1-p(i,j)$ and $p(i,j) < 1-p_{\text{add}}^u$. Besides, $1-2p_{\text{del}}^a < 1-2p_{\text{add}}^a$, therefore Eq. 11 holds.

Eq. 5 is equivalent to $\frac{1-U^u}{1-R} < l-2l\frac{1-U^a}{R} \Rightarrow \frac{Rp_{\text{del}}^u+(1-R)p_{\text{add}}^u}{1-R} < l-2l\frac{Rp_{\text{del}}^a+(1-R)p_{\text{add}}^a}{R} \Rightarrow \frac{(1-R)p_{\text{add}}^u}{1-R} < l-2l\frac{Rp_{\text{del}}^a}{R} \Rightarrow p_{\text{add}}^u < l(1-2p_{\text{del}}^a) \Rightarrow p_{\text{add}}^u < l(\frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a})$. Thus, we have

$$\begin{aligned} \frac{1}{p_{\text{add}}^u} - 1 &> \frac{1}{l\frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a}} - 1 \\ &> \frac{1}{p(i,j)\frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a}} - 1 \\ &= \left(\frac{1}{p(i,j)} - \frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a}\right) \frac{1-2p_{\text{add}}^a}{1-2p_{\text{del}}^a} \\ &= \left(\frac{1}{p(i,j)} - 1\right) \frac{1-2p_{\text{add}}^a}{1-2p_{\text{del}}^a} \end{aligned} \quad (17)$$

Eq. 17 is equivalent to

$$(1-p_{\text{add}}^u)(1-2p_{\text{del}}^a)p(i,j) > p_{\text{add}}^u(1-2p_{\text{add}}^a)(1-p(i,j)) \quad (18)$$

Combining with $p_{\text{del}}^u < p_{\text{add}}^u$, we have Eq. 12 satisfied. Therefore, we have proved that $p_{\text{add}}^{ua}(i,j) < \frac{1}{2}$ and $p_{\text{del}}^{ua}(i,j) < \frac{1}{2}$ under situation (c).

Situation (d): under situation (d), we only need to consider Eq. 4 and Eq. 5.

Eq. 5 is equivalent to $\frac{1-U^u}{1-R} < l(1 - 2\frac{1-U^a}{R}) \Rightarrow \frac{1-U^u}{1-R} < l \Rightarrow \frac{Rp_{\text{del}}^u + (1-R)p_{\text{add}}^u}{1-R} < l \Rightarrow p_{\text{add}}^u < l$. Therefore, we can obtain $p_{\text{add}}^u < p(i, j)$ and $1 - p(i, j) < 1 - p_{\text{del}}^u$. In addition, we have $1 - 2p_{\text{del}}^a > 1 - 2p_{\text{add}}^a$. Thus, Eq. 12 holds.

Eq. 4 is equivalent to $\frac{1-U^u}{1-R} < (1-h)(1 - 2\frac{1-U^a}{1-R}) \Rightarrow \frac{Rp_{\text{del}}^u + (1-R)p_{\text{add}}^u}{1-R} < (1-h)(1 - 2\frac{Rp_{\text{del}}^a + (1-R)p_{\text{add}}^a}{1-R}) \Rightarrow \frac{(1-R)p_{\text{add}}^u}{1-R} < (1-h)(1 - 2\frac{(1-R)p_{\text{add}}^a}{1-R}) \Rightarrow p_{\text{add}}^u < (1-h)(1 - 2p_{\text{add}}^a) \Rightarrow p_{\text{add}}^u < (1-h)(\frac{1-2p_{\text{add}}^a}{1-2p_{\text{del}}^a})$. Therefore, we have

$$\begin{aligned} \frac{1}{p_{\text{add}}^u} - 1 &> \frac{1}{(1-h)\frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a}} - 1 \\ &> \frac{1}{(1-p(i, j))\frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a}} - 1 \\ &= \left(\frac{1}{(1-p(i, j))} - \frac{1-2p_{\text{del}}^a}{1-2p_{\text{add}}^a} \right) \frac{1-2p_{\text{add}}^a}{1-2p_{\text{del}}^a} \\ &= \left(\frac{1}{(1-p(i, j))} - 1 \right) \frac{1-2p_{\text{add}}^a}{1-2p_{\text{del}}^a} \end{aligned} \quad (19)$$

Eq. 19 is equivalent to

$$(1 - p_{\text{add}}^u)(1 - 2p_{\text{del}}^a)(1 - p(i, j)) > p_{\text{add}}^u(1 - 2p_{\text{add}}^a)p(i, j) \quad (20)$$

Besides, $p_{\text{del}}^u < p_{\text{add}}^u$, therefore Eq. 11 is satisfied. Finally, we have proved that $p_{\text{add}}^{ua}(i, j) < \frac{1}{2}$ and $p_{\text{del}}^{ua}(i, j) < \frac{1}{2}$ under situation (d).

5.3 Achieving Successful De-anonymization

Since k is the number of incorrect mappings in $\sigma_k \neq \sigma_0$, $2 \leq k \leq n$ is satisfied. With σ_k , we consider $V_k^u \subseteq V^u$ as the set of incorrectly de-anonymized nodes, and $E_k^u = \{e_{i,j}^u | i \in V_k^u \text{ or } j \in V_k^u\}$ as the set of all the possible edges adjacent to at least one user in V_k^u , $E_\tau^u = \{e_{i,j}^u | i, j \in V_k^u, (i, j) \in E_k^u, \text{ and } (j, i) \in E_k^u\}$ as the set of all the possible edges corresponding to transposition mappings in σ_k , and $E^u = \{e_{i,j}^u | 1 \leq i \neq j \leq n\}$ as the set of all the possible edges on V . Furthermore, define $m_k = |E_k^u|$ and $m_\tau = |E_\tau^u|$. We have $|V_k^u| = k$, $m_k = \binom{k}{2} + k(n-k)$, $m_\tau \leq \frac{k}{2}$ since there are at most $\frac{k}{2}$ transposition mappings in σ_k , $|E^u| = \binom{n}{2}$.

Next, we quantify Φ_{σ_0} from a stochastic perspective. To quantify Φ_{σ_0} , we consider the DCN caused by the projection of each link including both the existing links and the non-existing links in the conceptually underlying graph, i.e., $\forall e_{i,j}^u \in E^u$. We further define $\Phi_{\sigma_k, E'}$ as the DCN caused by edges in the set E' under the mapping σ_k . If this link exists in G^u but not exist in G^a , according to the definition of DCN, it will cause a DCN of 2 and vice versa. Therefore, $\Phi_{\sigma_k} = \Phi_{\sigma_k, E^u \setminus E_k^u} + \Phi_{\sigma_k, E_k^u \setminus E_\tau^u} + \Phi_{\sigma_k, E_\tau^u}$ and $\Phi_{\sigma_0} = \Phi_{\sigma_0, E^u \setminus E_k^u} + \Phi_{\sigma_0, E_k^u \setminus E_\tau^u} + \Phi_{\sigma_0, E_\tau^u}$. Since $\Phi_{\sigma_k, E^u \setminus E_k^u} = \Phi_{\sigma_0, E^u \setminus E_k^u}$ and $\Phi_{\sigma_k, E_\tau^u} = \Phi_{\sigma_0, E_\tau^u}$, we can obtain $Pr(\Phi_{\sigma_k} \geq \Phi_{\sigma_0}) = Pr(\Phi_{\sigma_k, E_k^u \setminus E_\tau^u} \geq \Phi_{\sigma_0, E_k^u \setminus E_\tau^u})$.

Then, $\forall e_{i,j}^u \in E_k^u \setminus E_\tau^u$ under σ_k , it will be mapped to some other possible edge $\sigma_k(e_{i,j}^u) = e_{\sigma_k(i), \sigma_k(j)}^u \in E^u$ since $e_{i,j}^u \notin E_\tau^u$ and at least one of i and j is

incorrectly de-anonymized under σ_k . Therefore, in this case, the DCN/2 caused by $e_{i,j}^u$ during the projection process satisfies the Bernoulli distribution $\Phi_{\sigma_k, e_{i,j}^u} \sim B(1, p(i, j)^u (p(\sigma_k(i), \sigma_k(j))^u \times p_{\text{del}}^{ua}(i, j) + (1 - p(\sigma_k(i), \sigma_k(j))^u) \times (1 - p_{\text{add}}^{ua}(i, j)))) + (1 - p(i, j)^u) (p(\sigma_k(i), \sigma_k(j))^u \times (1 - p_{\text{del}}^{ua}(i, j)) + (1 - p(\sigma_k(i), \sigma_k(j))^u) \times p_{\text{add}}^{ua}(i, j)))$.

For $\sigma_0, \forall e_{i,j}^u \in E_k^u \setminus E_\tau^u$, the DCN/2 caused by $e_{i,j}^u$ after the projection process satisfies Bernoulli distribution $\Phi_{\sigma_0, e_{i,j}^u} \sim B(1, p(i, j)^u p_{\text{del}}^{ua} + (1 - p(i, j)^u) p_{\text{add}}^{ua})$.

Let $\lambda_{\sigma_0, e_{i,j}^u}$ and $\lambda_{\sigma_k, e_{i,j}^u}$ be the mean of $\Phi_{\sigma_0, e_{i,j}^u}$ and $\Phi_{\sigma_k, e_{i,j}^u}$, respectively. Since $p_{\text{add}}^{ua}(i, j) < \frac{1}{2}$ and $p_{\text{del}}^{ua}(i, j) < \frac{1}{2}$, we have $p_{\text{del}}^{ua}(i, j) < 1 - p_{\text{add}}^{ua}(i, j)$ and $p_{\text{add}}^{ua}(i, j) < 1 - p_{\text{del}}^{ua}(i, j)$. Furthermore, we can prove

$$\lambda_{\sigma_k, e_{i,j}^u} > p(i, j)^u p_{\text{del}}^{ua} + (1 - p(i, j)^u) p_{\text{add}}^{ua} = \lambda_{\sigma_0, e_{i,j}^u} \quad (21)$$

Lemma 3 [9] (i) Let $X \sim B(n_1, p)$ and $Y \sim B(n_2, p)$ be independent binomial variables. Then, $X + Y$ is also a binomial variable and $X + Y \sim B(n_1 + n_2, p)$; (ii) Let X and Y be two binomial random variables with means λ_x and λ_y , respectively. Then, when $\lambda_x > \lambda_y$, $Pr(X - Y \leq 0) \leq 2 \exp(\frac{(\lambda_x - \lambda_y)^2}{-8(\lambda_x + \lambda_y)})$.

By applying Lemma 3, we thus have

$$\begin{aligned} Pr(\Phi_{\sigma_k, e_{i,j}^u} > \Phi_{\sigma_0, e_{i,j}^u}) &> 1 - 2 \exp\left(-\frac{(\lambda_{\sigma_k, e_{i,j}^u} - \lambda_{\sigma_0, e_{i,j}^u})^2}{8\lambda_{\sigma_k, e_{i,j}^u} \lambda_{\sigma_0, e_{i,j}^u}}\right) \\ &= 1 - 2 \exp(-f(p(i, j)^u, p(\sigma_k(i), \sigma_k(j))^u) m_2) \end{aligned} \quad (22)$$

where $p(\cdot, \cdot)$ is a function of $p(i, j)^u$ and $p(\sigma_k(i), \sigma_k(j))^u$.

Similarly, we have $Pr(\Phi_{\sigma_k, E_k^u \setminus E_\tau^u} \geq \Phi_{\sigma_0, E_k^u \setminus E_\tau^u}) = Pr(\sum_{e_{i,j} \in E_k^u \setminus E_\tau^u} \Phi_{\sigma_k, e_{i,j}^u} \geq \sum_{e_{i,j} \in E_k^u \setminus E_\tau^u} \Phi_{\sigma_0, e_{i,j}^u}) \geq \prod_{e_{i,j} \in E_k^u \setminus E_\tau^u} Pr(\Phi_{\sigma_k, e_{i,j}^u} \geq \Phi_{\sigma_0, e_{i,j}^u})$.

Considering $f_{\min} = \min f(p(i, j), p(\sigma_k(i), \sigma_k(j)))$, we can obtain $Pr(\Phi_{\sigma_k, E_k^u \setminus E_\tau^u} \geq \Phi_{\sigma_0, E_k^u \setminus E_\tau^u}) \geq \prod_{e_{i,j} \in E_k^u \setminus E_\tau^u} Pr(\Phi_{\sigma_k, e_{i,j}^u} \geq \Phi_{\sigma_0, e_{i,j}^u}) \geq \prod_{e_{i,j} \in E_k^u \setminus E_\tau^u} (1 - 2 \exp(-f_{\min} m_2)) = (1 - 2 \exp(-f_{\min} m_2))^{m_k}$.

Lemma 4

$$\lim_{x \rightarrow \infty} (1 - \exp(-(ax + b)))^{cx+d} = 1 \quad (23)$$

where a, b, c, d are positive real numbers.

Proof. Let us take logarithm of Eq.(23), it is thus equivalent to $\lim_{x \rightarrow \infty} (cx + d) \ln(1 - 2 \exp(-(ax + b))) = 0$. Then, we can prove that

$$\begin{aligned} &\lim_{x \rightarrow \infty} (cx + d) \ln(1 - 2 \exp(-(ax + b))) \\ &= \lim_{x \rightarrow \infty} \frac{\ln(1 - 2 \exp(-(ax + b)))}{\frac{1}{cx+d}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{2a \exp(-(ax+b))}{1 - 2 \exp(-(ax+b))}}{\frac{-c}{(cx+d)^2}} \quad (L'Hospital's Rule) \\ &= \lim_{x \rightarrow \infty} \frac{c}{-2a^3 \exp(ax + b)} \\ &= 0. \end{aligned} \quad (24)$$

Applying Lemma 4, we obtain the final result of Theorem 1 as

$$\lim_{n \rightarrow \infty} Pr(\Phi_{\sigma_k, E_k^u \setminus E_r^u} \geq \Phi_{\sigma_0, E_k^u \setminus E_r^u}) = 1 \quad (25)$$

5.4 Relation Between the Global Structure Utility and De-anonymization Capability

In Definition 1, we consider a straightforward local neighborhood utility metric, which evaluates the distortion between the *adjacency matrices* of the two graphs, i.e., $\|\mathbf{A}_a - \mathbf{A}\|_1$. However, the real-world data utility is application-oriented such that we need to consider a more general utility metric, to incorporate more aggregate information of the graph instead of just the *adjacency matrix*. Motivated by the general utility distance in [31, 32], we consider to utilize the w -th power of the transition probability matrix T^w , which is induced by the w -hop random walk on graph G , to define the global structure utility as follows:

Definition 2 *The global structure utility for the anonymized graph G_a is defined as*

$$U_{a(w)} = 1 - \frac{\|\mathbf{T}_a^w - \mathbf{T}^w\|_1}{2N} \quad (26)$$

where $\mathbf{T}_a^w, \mathbf{T}^w$ are the w -th power of the transition probability matrix \mathbf{T}_a, \mathbf{T} , respectively. The denominator in Eq. 26 is a normalization factor to guarantee $U_{a(w)} \in [0, 1]$. Similarly, the global structure utility for the auxiliary graph is

$$U_{u(w)} = 1 - \frac{\|\mathbf{T}_u^w - \mathbf{T}^w\|_1}{2N} \quad (27)$$

Our metric of global structure utility in Definition 2 is intuitively reasonable for a broad class of real-world applications, and captures the w -hop random walks between the conceptually underlying graph G and the anonymized graph G_a . We note that random walks are closely linked to structural properties of real-world data. For example, a lot of high-level social network based applications such as recommendation systems [33], Sybil defenses [34] and anonymity systems [35] directly perform random walks in their protocols. The parameter w is application specific; for applications that require access to fine-grained community structure, such as recommendation systems [33], the value of w should be small (typically 2 or 3). For other applications that utilize coarse and macro community structure of the data, such as Sybil defense mechanisms [34], w can be set to a larger value (typically around 10). Therefore, our global structure utility metric can quantify the utility performance of a perturbed graph for various real-world applications in a general and universal manner.

Based on this general utility metric, we further theoretically analyze the de-anonymization capability of structure-based attacks and quantify the anonymized utility for successful de-anonymization.

Theorem 2 *For any $\sigma_k \neq \sigma_0$, where k is the number of incorrectly-mapped nodes between G_a and G_u , $\lim_{n \rightarrow \infty} Pr(\Phi_{\sigma_k} \geq \Phi_{\sigma_0}) = 1$ when the following conditions are satisfied:*

$$U_{u(w)} + 2lU_{a(w)} > 1 + 2l - \frac{wRl(N-1)}{2} \quad (28)$$

$$U_{u(w)} + 2(1-h)U_{a(w)} > 1 + 2(1-h) - \frac{w(N-1)(1-h)(1-R)}{2} \quad (29)$$

$$U_{u(w)} + 2l\frac{1-R}{R}U_{a(w)} > 1 + 2l\frac{1-R}{R} - \frac{wl(1-R)(N-1)}{2} \quad (30)$$

$$U_{u(w)} + 2(1-h)\frac{R}{1-R}U_{a(w)} > 1 + 2(1-h)\frac{R}{1-R} - \frac{wR(1-h)(N-1)}{2} \quad (31)$$

Proof. We first relate the adjacency matrix \mathbf{A} with the transition probability matrix \mathbf{T} as $\mathbf{A} = \mathbf{\Lambda}\mathbf{T}$, where $\mathbf{\Lambda}$ is a diagonal matrix and $\mathbf{\Lambda}(i, i) = \text{deg}(i)$. Then we analyze the global utility distance for the anonymized graph. When $w = 1$, we can prove $\|\mathbf{A}_a - \mathbf{A}\|_1 = \|\mathbf{\Lambda}_a\mathbf{T}_a - \mathbf{\Lambda}\mathbf{T}\|_1 = \|(\mathbf{\Lambda}_a\mathbf{T}_a - \mathbf{\Lambda}_a\mathbf{T} + \mathbf{\Lambda}_a\mathbf{T} - \mathbf{\Lambda}\mathbf{T})\|_1 \geq \|\mathbf{\Lambda}_a\|_1\|\mathbf{T}_a - \mathbf{T}\|_1$. Since the element in the diagonal of $\mathbf{\Lambda}_a$ is greater than 1, we have $\|\mathbf{A}_a - \mathbf{A}\|_1 \geq \|\mathbf{T}_a - \mathbf{T}\|_1$. Therefore, we can obtain $U_a \leq U_{a(w)}$. Similarly, we have $U_u \leq U_{u(w)}$. Incorporating these two inequalities into Eqs. 3,4,5,6, we have Theorem 2 satisfied under $w = 1$. Next, we consider $w \geq 1$. It is easy to prove that $\|\mathbf{T}_a^w - \mathbf{T}^w\|_1 \leq w\|\mathbf{T}_a - \mathbf{T}\|_1$ so $\|\mathbf{T}_a^w - \mathbf{T}^w\|_1 \leq w\|\mathbf{A}_a - \mathbf{A}\|_1$. Therefore, we have $U_a \leq wU_{a(w)} + 1 - w$. Similarly, we also have $U_u \leq wU_{u(w)} + 1 - w$ for the auxiliary graph. Incorporating these two inequalities into Eqs. 3,4,5,6, we thus have Theorem 2 proved.

Similar to Theorem 1, when the global structure utility for the anonymized graph and the auxiliary graph satisfies all of the four conditions in Theorem 2, we can achieve successful de-anonymization from a statistical perspective. With rather high probability, the attacker can find out the correct mapping between the anonymized graph and the auxiliary graph, by choosing the mapping with the minimal DCN out of all the potential mappings.

Furthermore, both Theorem 1 and Theorem 2 give meaningful guidelines for future designs of de-anonymization and anonymization schemes: 1) Since successful de-anonymization is theoretically achievable when the anonymized utility satisfies the conditions in Theorem 1 (for the local neighborhood utility) and Theorem 2 (for the global structure utility), the gap between the practical de-anonymization accuracy and the theoretically achievable performance can be utilized to evaluate the effectiveness of a real-world de-anonymization attack; 2) we can also leverage Theorem 1 and Theorem 2 for designing future secure data publishing to defend against de-anonymization attacks. For instance, a secure data publishing scheme should provide anonymized utility that locates out of the theoretical bound (green shaded region) in Figure 1 while enabling real-world applications. We will provide a practical analysis for such privacy and utility tradeoffs in Section 6.

6 Practical Privacy and Utility Trade-off

In this section, we show how the theoretical analysis in Section 4 can be utilized to evaluate the privacy risks of practical data publishing and the performance of

Table 1. De-anonymization *Accuracy* of State-of-the-Art De-anonymization Methods.

Datasets	Method	<i>noise</i> = 0.05	<i>noise</i> = 0.15	<i>noise</i> = 0.25
Facebook	Ji et al. [9]	0.95	0.81	0.73
Facebook	Nilizadeh et al. [10]	0.83	0.74	0.68
Collaboration	Ji et al. [9]	0.27	0.09	0.05
Collaboration	Nilizadeh et al. [10]	0.58	0.23	0.11
Twitter: small	Ji et al. [9]	0.55	0.39	0.21
Twitter: small	Nilizadeh et al. [10]	0.92	0.79	0.69
Twitter: large	Ji et al. [9]	0.48	0.21	0.12
Twitter: large	Nilizadeh et al. [10]	0.91	0.66	0.21

practical de-anonymization attacks. To enable real-world applications without compromising the privacy of users, a secure data anonymization scheme should provide anonymized utility which does not locate within the utility region for perfect de-anonymization shown as the green shaded regions in Figure 1. From a data publisher’s point of view, we consider the worst-case attacker who has access to perfect auxiliary information, i.e., $noise_u = 0$. Based on Theorem 1, we aim to quantify the amount of noise that is added to the anonymized data for achieving successful de-anonymization. Note that our derivation is from a statistical point of view instead of from the perspective of a concrete graph.

Theorem 1. *When the noise of the anonymized graph is less than 0.25, successful de-anonymization can be theoretically achieved.*

Proof. For the anonymization method of Hay et al. in [2], we have $P_a^{del} = k_a/M_a$ and $P_a^{add} = k_a/(\binom{N}{2} - M_a)$. Similarly, we have $P_u^{del} = k_u/M_u$ and $P_u^{add} = k_u/(\binom{N}{2} - M_u)$. Based on our utility metric in Definition 1, we have $U_u = 1 - 2R \times noise_u$ and $U_a = 1 - 2R \times noise_a$. Considering the sparsity property in most real-world structural graphs [7], the utility condition for achieving successful de-anonymization is restricted by Eq. 3, which can be represented as $noise_u + l \times noise_a < \frac{l}{2}$. Consider the worst-case attacker who has access to perfect auxiliary information, i.e., $noise_u = 0$. Therefore, we have $noise_a < 0.25$.

Therefore, when the *noise* added to the anonymized graph is less than 0.25, there would be a serious privacy breach since successful de-anonymization is theoretically achievable. Note that such a utility bound only conservatively provides the minimum noise that should be added to the anonymized data. Practically, we suggest a real-world data publisher to add more noise to protect the privacy of the data. Furthermore, such privacy-utility trade-off can be leveraged as a guide for designing new anonymization schemes.

In addition, our derived theoretical analysis can also be utilized to evaluate the performance of existing de-anonymization attacks. We implement our experiments on Facebook dataset, Collaboration dataset [36], Twitter dataset [10]. The Facebook dataset [37] which contains 46,952 nodes (i.e., users) connected by 876,993 edges (i.e., social relationships). The Collaboration dataset [36] is a network of co-authorships between scientists who have posted preprints on the

Condensed Matter E-Print Archive, which consists of $36,458$ users and $171,735$ edges. The Twitter dataset [10] captures the connections between users who mentioned each other at least once between March 24th, 2012 and April 25th, 2012, and contains two different graphs named Twitter (small) with $9,745$ users and $50,164$ edges, and Twitter (large) with $90,331$ users and $358,422$ edges.

To evaluate the performance of existing de-anonymization attacks, we consider a popular perturbation method of Hay et al. in [2], which applies a sequence of r edge deletions followed by r random edge insertions. A similar perturbation process has been utilized for the de-anonymization attacks in [10]. Candidates for edge deletion are sampled uniformly at random from the space of the existing edges in graph G , while candidates for edge insertion are sampled uniformly at random from the space of edges that are not existing in G . Here, we define *noise* (perturbations) as the extent of edge modification, i.e., the ratio of altered edges r to the total number of edges, i.e., $noise = \frac{r}{M}$. Note that we add the same amount of *noise* to the original graph to obtain the anonymized graph and the auxiliary graph, respectively. Then, we apply the state-of-the-art de-anonymization attacks in [9] and [10] to de-anonymize the anonymized graph by leveraging the auxiliary graph.

We utilize *Accuracy* as an effective evaluation metric to measure the de-anonymization performance. *Accuracy* is the ratio of the correctly de-anonymized nodes out of all the overlapped nodes between the anonymized graph and the auxiliary graph:

$$Accuracy = \frac{N_{cor}}{|V_a \cap V_u|}, \quad (32)$$

where N_{cor} is the number of correctly de-anonymized nodes. The *Accuracy* of these de-anonymization attacks corresponding to different levels of *noise* is shown in Table 1.

From Table 1, we can see that the state-of-the-art de-anonymization attacks can only achieve less than 75% de-anonymization accuracy when the *noise* is 0.25, which demonstrates the ineffectiveness of previous work and the potential of developing more powerful de-anonymization attacks in the future.

7 DISCUSSION

There is a clear trade-off between utility and privacy for data publishing. In this work, we analytically quantify the relationships between the utility of anonymized data and the de-anonymization capability. Our quantification results show that privacy could be breached if the utility of anonymized data is high. Hence, striking the balance between utility and privacy for data publishing is important yet difficult - providing the high utility for real-world applications would decrease the data's resistance to de-anonymization attacks.

Suggestions for Secure Data Publishing. Secure data publishing (sharing) is important for companies (e.g., online social network providers), governments and researchers. Here, we give several general guidelines: (i) Data owners should carefully evaluate the potential vulnerabilities of the data before publishing. For

example, our quantification result in Section 4 can be utilized to evaluate the vulnerabilities of the structural data. (ii) Data owners should develop proper policies on data collections to defend against adversaries who aim to leverage auxiliary information to launch de-anonymization attacks. To mitigate such privacy threats, online social network providers, such as Facebook, Twitter, and Google+, should reasonably limit the access to users' social relationships.

8 CONCLUSION

In this paper, we theoretically analyze the de-anonymization attacks and provide conditions on the utility of the anonymized data (denoted by anonymized utility) to achieve successful de-anonymization under a general graph model. Our analysis provides a theoretical foundation for structure-based de-anonymization attacks, and can serve as a guide for designing new de-anonymization/anonymization systems in practice. By comparing these experimental results and the theoretically achievable de-anonymization capability derived in our analysis, future work can include studying our utility versus privacy trade-offs for more datasets, and designing more powerful anonymization/de-anonymization approaches.

References

1. W.-H. Lee, C. Liu, S. Ji, P. Mittal, and R. B. Lee, "Quantification of de-anonymization risks in social networks," in *Information Systems Security and Privacy (ICISSP)*. SCITEPRESS, 2017.
2. M. Hay, G. Miklau, D. Jensen, P. Weis, and S. Srivastava, "Anonymizing social networks," *Computer Science Department Faculty Publication Series*, 2007.
3. K. Liu and E. Terzi, "Towards identity anonymization on graphs," in *SIGMOD*, 2008.
4. P. Pedarsani and M. Grossglauer, "On the privacy of anonymized networks," in *SIGKDD*, 2011.
5. M. Srivatsa and M. Hicks, "Deanonymizing mobility traces: Using social network as a side-channel," in *CCS*, 2012.
6. L. Backstrom, C. Dwork, and J. Kleinberg, "Wherefore art thou r3579x?: anonymized social networks, hidden patterns, and structural steganography," in *WWW*, 2007.
7. A. Narayanan and V. Shmatikov, *Robust de-anonymization of large sparse datasets*. IEEE S&P, 2008.
8. ———, *De-anonymizing social networks*. IEEE S&P, 2009.
9. S. Ji, W. Li, M. Srivatsa, and R. Beyah, "Structural data de-anonymization: Quantification, practice, and implications," in *CCS*, 2014.
10. S. Nilizadeh, A. Kapadia, and Y.-Y. Ahn, "Community-enhanced de-anonymization of online social networks," in *CCS*, 2014.
11. W.-H. Lee, C. Liu, S. Ji, P. Mittal, and R. Lee, "Blind de-anonymization attacks using social networks," in *Proceedings of the 16th Workshop on Privacy in the Electronic Society*. ACM, 2017.
12. M. Hay, G. Miklau, D. Jensen, D. Towsley, and P. Weis, "Resisting structural re-identification in anonymized social networks," *VLDB Endowment*, 2008.

13. K. Sharad and G. Danezis, "An automated social graph de-anonymization technique," in *Proceedings of the 13th Workshop on Privacy in the Electronic Society*. ACM, 2014.
14. —, "De-anonymizing d4d datasets," in *Workshop on Hot Topics in Privacy Enhancing Technologies*, 2013.
15. F. Buccafurri, G. Lax, A. Nocera, and D. Ursino, "Discovering missing me edges across social networks," *Information Sciences*, 2015.
16. A. Machanavajjhala, D. Kifer, J. Gehrke, and M. Venkitasubramaniam, "l-diversity: Privacy beyond k-anonymity," *ACM Transactions on Knowledge Discovery from Data*, 2007.
17. C. Dwork, "Differential privacy," in *Encyclopedia of Cryptography and Security*. Springer, 2006.
18. C. Liu, S. Chakraborty, and P. Mittal, "Dependence makes you vulnerable: Differential privacy under dependent tuples," in *NDSS*, 2016.
19. A. Sala, X. Zhao, C. Wilson, H. Zheng, and B. Y. Zhao, "Sharing graphs using differentially private graph models," in *IMC*, 2011.
20. J. Xie, S. Kelley, and B. K. Szymanski, "Overlapping community detection in networks: The state-of-the-art and comparative study," *Acm computing surveys (csur)*, vol. 45, no. 4, p. 43, 2013.
21. P. Pedarsani, D. R. Figueiredo, and M. Grossglauser, "A bayesian method for matching two similar graphs without seeds." in *Allerton*, 2013.
22. S. Ji, W. Li, P. Mittal, X. Hu, and R. Beyah, "Secgraph: A uniform and open-source evaluation system for graph data anonymization and de-anonymization," in *USENIX Security Symposium*, 2015.
23. P. Erdős and A. Rényi, "On the evolution of random graphs," *Selected Papers of Alfréd Rényi*, 1976.
24. M. Newman, *Networks: an introduction*. Oxford University Press, 2010.
25. M. E. Newman, "The structure and function of complex networks," *SIAM review*, 2003.
26. C. Fabiana, M. Garetto, and E. Leonardi, "De-anonymizing scale-free social networks by percolation graph matching," in *INFOCOM*, 2015.
27. S. Ji, W. Li, N. Z. Gong, P. Mittal, and R. Beyah, "On your social network de-anonymizability: Quantification and large scale evaluation with seed knowledge," in *NDSS*, 2015.
28. N. Korula and S. Lattanzi, "An efficient reconciliation algorithm for social networks," *Proceedings of the VLDB Endowment*, 2014.
29. H. Pham, C. Shahabi, and Y. Liu, "Ebm: an entropy-based model to infer social strength from spatiotemporal data," in *SIGMOD*, 2013.
30. R. W. Hamming, "Error detecting and error correcting codes," *Bell System technical journal*, 1950.
31. P. Mittal, C. Papamanthou, and D. Song, "Preserving link privacy in social network based systems," in *NDSS*, 2013.
32. C. Liu and P. Mittal, "Linkmirage: Enabling privacy-preserving analytics on social relationships," in *NDSS*, 2016.
33. R. Andersen, C. Borgs, J. Chayes, U. Feige, A. Flaxman, A. Kalai, V. Mirrokni, and M. Tennenholtz, "Trust-based recommendation systems: an axiomatic approach," in *WWW*, 2008.
34. H. Yu, P. B. Gibbons, M. Kaminsky, and F. Xiao, "Sybillimit: A near-optimal social network defense against sybil attacks," in *IEEE S&P*, 2008.
35. P. Mittal, M. Wright, and N. Borisov, "Pisces: Anonymous communication using social networks," *NDSS*, 2013.

36. M. E. Newman, "The structure of scientific collaboration networks," *Proceedings of the National Academy of Sciences*, 2001.
37. B. Viswanath, A. Mislove, M. Cha, and K. P. Gummadi, "On the evolution of user interaction in facebook," in *ACM workshop on Online social networks*, 2009.